# Entropic Fluctuating Hydrodynamics for Anomalous Scaling in Low-Dimensional Heat Conduction

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## Abstract

Scaling relations between anomalous signatures in low-dimensional thermal transport, *i.e.*, length-dependence and non-Brownian growth, call for a universal interpretation. An entropic Burgers-Kardar-Parisi-Zhang class is derived based on fluctuations of stochastic entropy production in heat conduction. A generic scaling framework of the anomalous thermal conductivity is thereafter established from the dynamical scaling theory of fluctuating hydrodynamics. This scaling framework demonstrates that two sound scaling laws between the length-dependent thermal conductivity and mean square of displacement are traceable to Galilean invariance. It also expects other anomalous behaviors including the deviation between equilibrium and non-equilibrium schemes, dimensionality-dependence, logarithmic divergence, failure of the "transit time" truncation, and non-Kardar-Parisi-Zhang scaling. Our results and their coincidences with existing investigations illustrate that Langevin-like formalisms for stochastic thermodynamics can be applied to energy transport in low-dimensional and mesoscopic systems.

## 1. Introduction

Classical Fourier's law of heat conduction, $\boldsymbol{J}_Q = -\kappa \nabla T$, wherein $\boldsymbol{J}_Q = \boldsymbol{J}_Q(\boldsymbol{x}, t)$ is the local heat flux, $\kappa$ is the thermal conductivity and $T = T(\boldsymbol{x}, t)$ is the local temperature, has been proved by numerous experiments in three-dimensional (3D) bulk materials. In low-dimensional systems, the validity of Fourier's law remains open and has attracted intensive attention nowadays.[1-5] In a paradigmatic review,[4] anomalous behaviors in low-dimensional thermal transport have been classified as five signatures including length-dependent thermal conductivity, power-law decay of the heat current correlation function, non-Brownian mean square of displacement (MSD), fast relaxation of spontaneous fluctuations, and nonlinearity of stationary temperature profiles. The length-dependence, namely, $\kappa = \kappa(L)$ with $L$ the system length, is the most widely discussed. In one-dimensional (1D) and quasi-one-dimensional systems, the length-dependence obeys power-law, $\kappa \sim L^\alpha$, while the logarithmic divergence, $\kappa \sim \ln L$, is usually observed in two

dimensional (2D) cases. Either power-law or logarithmic divergence indicates a sharp enhancement of heat transport with increasing lengths and surprisingly high thermal conductivity at macroscale, which are intriguing for engineering. Recent advances on the length-dependence of anomalous heat conduction in low-dimensional nano-materials can be found in Ref. [6], which also summarized the underlying fundamental physics as well as a perspective view for thermal controlling and management.

The length-dependent thermal conductivity is often connected to other anomalous signatures such as the power-law decay like $C(t) \sim t^{\eta-1}$ and non-Brownian MSD $\langle \Delta|\boldsymbol{x}|^2 \rangle \sim t^\gamma$, where $C(t)$ stands for the heat current autocorrelation function, $\eta \leq 1$ and $0 < \gamma \leq 2$. The connection to the power-law decay arises from the following Green-Kubo formula,

$$\kappa = \lim_{L \to +\infty} \frac{1}{k_B T^2 L} \int_0^{t_C} C(t) dt, \qquad (1)$$

where $k_B$ denotes the Boltzmann constant and $t_C$ is a cutoff time. One common choice of $t_C$ is the so-called "transit time", namely, $t_C = \frac{L}{v}$ with $v$ the sound velocity. Then, one can acquire $\alpha = \eta$ for $\eta \neq 0$ and $\kappa \sim \ln L$ for $\eta = 0$. There exist connections between the length-dependence exponent $\alpha$ and growth exponent of the MSD $\gamma$ as well. Denisov *et al.*[7]

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demonstrated $\alpha = \gamma - 1$ in the noninteracting Lévy walk (LW) model. This result is later recovered in Ref. [8], which does not assume any specific random walk model but relies on the local continuity equation. Based on the length-dependence of the mean first passage time (MFPT) $\mathcal{T}_{MFPT}$, Li and Wang[9] acquired a different scaling relation in billiard gas channel models, $\alpha = 2 - \frac{2}{\gamma}$. Both $\alpha = \gamma - 1$ and $\alpha = 2 - \frac{2}{\gamma}$ have been supported by numerical simulations.[10-15] In Ref. [8], $\alpha = \gamma - 1$ is universal from the equilibrium Green-Kubo method, whereas $\alpha = 2 - \frac{2}{\gamma}$ is in the non-equilibrium scheme, namely, $\kappa = -\frac{(J_Q L)}{\Delta T}$ with $\Delta T = T(x + L) - T(x)$ denoting the temperature difference.

Several spontaneous problems then occur: (i) whether the two scaling laws can be unified in a universal formalism; (ii) the dimensionality effects are not explicitly reflected in the scaling laws; and (iii) the logarithmic divergence is not expected in the scaling laws. $\alpha = \gamma - 1$ emerges from the relation $C(t) \propto \frac{d^2 \langle |\Delta x|^2 \rangle}{dt^2}$, and hence, $\kappa \propto \frac{d \langle |\Delta x|^2 \rangle}{dt}$ is always power-law even in the limit $\gamma \to 1^+$. As a result of $\mathcal{T}_{MFPT} \propto L^{\frac{2}{\gamma}}$ and $\kappa(L) \propto \frac{L^2}{\mathcal{T}_{MFPT}}$, $\alpha = 2 - \frac{2}{\gamma}$ cannot lead to the logarithmic divergence either. The main aim of the present work is to address these problems via a generic scaling framework.

## 2. Entropic fluctuating hydrodynamics and anomalous scaling

### 2.1. Entropic Burgers-Kardar-Parisi-Zhang class

Our approach is in the spirit of nonlinear fluctuating hydrodynamics, which has been applied to the divergence induced by momentum conservation[16] and is usually associated with the mode-coupling theory (MCT).[17] These previous studies follow the Landau-Lifshitz theory [18], which introduces noisy terms into the conventional Navier-Stokes equations. For pure heat conduction, it focuses on the temperature or energy fluctuations, which is governed by the linear Edwards-Wilkinson (EW) equation[19] as follows

$$\frac{\partial T_d}{\partial t} = D\nabla^2 T_d + \eta_T, \quad \langle \eta_T(x,t)\eta_T(x',t') \rangle \propto T_{eq}^2 \delta(x - x')\delta \quad (2)$$

In Eq. (2), $T_d = T - T_{eq}$ is the deviation from the equilibrium state $T_{eq}$, $D = \frac{\kappa}{c}$ denotes the thermal diffusivity, $c$ is the specific heat capacity per volume, and $\eta_T$ is the additive white Gaussian noise. Different from the Landau-Lifshitz theory, we consider fluctuations of stochastic entropy production $\Delta S_{tot}(t)$ rather than temperature, which is a key thermodynamic observable in mesoscopic systems. One celebrated result about the stochastic entropy production is that the second law in the framework of stochastic thermodynamics is stated by the nonnegativity of the mean value $\langle \Delta S_{tot}(t) \rangle$.[20,21] Other statistical features of $\Delta S_{tot}(t)$ like the infimum,[22] fluctuation theorems,[23,24] first-passage times,[25,26] and variance,[27] attracts increasing interest in recent years, which can be deduced from Langevin-like equations for

the entropy production rate. The simplest example[22] is a drift-diffusion process governed by

$$\frac{d\Delta S_{tot}(t)}{dt} = v_S + \eta_S(t), \quad (3)$$

where $v_S \geq 0$ is the entropic drift and $\eta_S$ stands for the noise term. In a sound work by Pigolotti *et al.*,[28] this equation is demonstrated as the steady-state situation of their Itô stochastic differential equation, which arises from coupled over-damped dynamics and can identify generic properties of the stochastic entropy production.

We now extend this formalism into the density of the stochastic entropy production rate $\sigma = \sigma(x,t)$. For heat conduction close to steady-state, the average rate of the stochastic entropy production agrees with the phenomenological expression of classical irreversible thermodynamics: [29,30]

$$\left\langle \frac{d\Delta S_{tot}(t)}{dt} \right\rangle \sim \int \kappa \frac{|\nabla T|^2}{T^2} d^d \boldsymbol{x} = \int \boldsymbol{J}_Q \nabla \left( \frac{1}{T} \right) d^d \boldsymbol{x}. \quad (4)$$

In the entropic Langevin equation, the drift velocity $v_S$ corresponds to the average entropy production rate. Due to this conceptual connection, the density of the stochastic entropy production rate is formulated as

$$\sigma = \sigma(\boldsymbol{x},t) = \kappa \frac{|\nabla T|^2}{T^2} + \eta_{SV}, \quad (5)$$

wherein $\eta_{SV}$ is an entropic noise in the volume element. Besides Eq. (5), $\sigma(\boldsymbol{x},t)$ should satisfy $\frac{dS_{tot}(t)}{dt} = \int \sigma d^d \boldsymbol{x}$ and be restricted by the entropy balance equation

$$\sigma = \frac{\partial s}{\partial t} + \nabla \boldsymbol{J}_S, \quad (6)$$

with $s = s(\boldsymbol{x},t)$ the entropy density and $\boldsymbol{J}_S = \boldsymbol{J}_S(\boldsymbol{x},t)$ the entropy flux. In classical irreversible thermodynamics [31], $s(\boldsymbol{x},t)$ and $\boldsymbol{J}_S(\boldsymbol{x},t)$ are given by

$$s(\boldsymbol{x},t) = \int^{T(x,t)} c \frac{dT}{T}, \quad (7)$$

$$\boldsymbol{J}_S(\boldsymbol{x},t) = \frac{J_Q(x,t)}{T(x,t)}. \quad (8)$$

Upon substituting Eqs. (5, 7, 8) into the entropy balance equation, we acquire the following governing equation for the entropy density:

$$\sigma = \frac{\partial s}{\partial t} + \nabla \boldsymbol{J}_S$$
$$\Rightarrow \kappa \frac{|\nabla T|^2}{T^2} + \eta_{SV} = \frac{\partial s}{\partial t} + \nabla \left[ \frac{\boldsymbol{J}_Q}{T} \right]$$
$$\Rightarrow \frac{\partial s}{\partial t} + \nabla \left[ -\frac{\kappa \nabla T}{T} \right] = \kappa \frac{|\nabla T|^2}{T^2} + \eta_{SV}$$
$$\Rightarrow \frac{\partial s}{\partial t} + \nabla \left[ -\frac{\kappa \nabla s}{c} \right] = \kappa \frac{|\nabla s|^2}{c^2} + \eta_{SV}$$
$$\Rightarrow \frac{\partial s}{\partial t} = D \left( \nabla^2 s + \frac{1}{c} |\nabla s|^2 \right) + \eta_{SV}. \quad (9)$$

If the entropic noise takes the form $\langle \eta_{SV}(\boldsymbol{x},t)\eta_{SV}(\boldsymbol{x}',t') \rangle \propto \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$, Eq. (9) becomes the well-known Kardar-Parisi-Zhang (KPZ) equation.[32] When the temperature evolution is considered, Eq. (9) is equivalent to adding a multiplicative correction to the Landau-Lifshitz theory. The derivation of Eq. (9) relies on small fluctuations so that the temperature-dependence can be ignored. In order to guarantee well-defined thermodynamic quantities, the evolutions must be sufficiently slow as well. We recall that Eq. (4) is

approximate and requires near steady-state. It only includes the adiabatic entropy production caused by the stationary irreversible current, but neglects non-adiabatic and transient housekeeping entropy production rates, which are produced by the non-stationary states and asymmetry of the probability density function (PDF), respectively. A careful research[30] unveils that the temporal derivative $\frac{\partial T}{\partial t}$ will also contribute to the average rate of stochastic entropy production.

Through taking the gradient of Eq. (9), one can obtain the stochastic Burgers equation for the entropy flux

$$c\frac{\partial J_S}{\partial t} + 2(J_S\nabla)J_S = \kappa(\nabla^2 J_S + \nabla\eta_{SV}). \tag{10}$$

Though the entropy flux performs similarly to a Newtonian fluid in Eq. (10), the underlying nature of the entropic Burgers equation is fundamentally different from the usual momentum equation. In Eq. (10), the nonlinear term $(J_S\nabla)J_S$ is induced by the entropy production rate,

$$(J_S\nabla)J_S = \nabla(J_S J_S) = \nabla\left(\frac{1}{\kappa}\langle\sigma\rangle\right). \tag{11}$$

Nevertheless, in the momentum equation of a Newtonian fluid, the nonlinear term is extraneous to irreversibility, which remains unchanged in the inviscid case. In the deterministic case of which $\eta_{SV} \equiv 0$, Eqs. (9) and (10) reduce to the entropic governing equations of macroscopic heat conduction.

The Burgers-KPZ class has been applied to various subjects like interface growth,[33] Burgers turbulence,[34] directed polymers,[35] coffee stains,[36] chemical reaction fronts,[37] polar active fluids,[38] and quantum entanglement growth.[39] Its self-consistent asymptotic behaviors are commonly described in terms of three scaling exponents $(\chi, z, \beta) \in \mathbb{R}^3$, where $\chi$ is the roughness exponent, $z$ is the dynamical exponent, and $\beta = \frac{\chi}{z}$ is the early-time exponent. For instance, the long-time asymptotics of the MSD fulfills $\langle|\Delta x|^2\rangle \sim t^\gamma$ with $\gamma = \frac{2}{z}$. Here, the Burgers-KPZ class is established for entropic evolutions of mesoscopic heat conduction, and in the following, it will be used to derive anomalies in both equilibrium and non-equilibrium schemes.

## 2.2. Equilibrium Green-Kubo method

We first discuss the Green-Kubo formula. In the limit $L \to +\infty$, the time-dependence of the correlation function $C(t)$ is dominated by the large scaling asymptotics of the heat flux, which can be estimated by the correlator $\lim_{|x|\to+\infty}\langle J_Q(x,t)J_Q(x,t=0)\rangle$ approximately. For sufficiently small temperature fluctuations, one can write

$$J_S = \frac{1}{T_{eq}}\left[1 + O\left(\frac{T_d}{T_{eq}}\right)\right]J_Q, \tag{12}$$

which yields

$$\frac{\langle J_Q(x,t)J_Q(x,t=0)\rangle}{T_{eq}^2} \approx \langle J_S(x,t)J_S(x,t=0)\rangle. \tag{13}$$

In the Burgers-KPZ class, the large scaling of the entropic correlator $\langle J_S(x,t)J_S(x,t=0)\rangle$ is formulated as[40]

$$\lim_{|x|\to+\infty}\langle J_S(x,t)J_S(x,t=0)\rangle \propto t^{2\left(\beta-\frac{1}{z}\right)}. \tag{14}$$

Thereupon, we have $\frac{C(t)}{T_{eq}^2} \propto t^{2\left(\beta-\frac{1}{z}\right)}$, which leads to

$$\kappa \propto \begin{cases} t_C^{2\left(\beta-\frac{1}{z}\right)+1}, 2\left(\beta-\frac{1}{z}\right) \neq -1 \\ ln\, t_C, 2\left(\beta-\frac{1}{z}\right) = -1 \end{cases}. \tag{15}$$

As the "transit time" truncation is used, we arrive at

$$\alpha = 2\left(\beta-\frac{1}{z}\right) + 1, 2\left(\beta-\frac{1}{z}\right) \neq -1, \tag{16}$$

and $\kappa \sim ln\, L$ for $2\left(\beta-\frac{1}{z}\right) = -1$. The scaling exponents usually depend on the system dimensionality $d$, and dimensionality-dependence of the thermal conductivity therefore occurs. In 1D systems, the KPZ universality class reads $(\chi, z, \beta) = \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{3}\right)$, which indicates $\alpha = \frac{1}{3}$. The logarithmic divergence in 2D systems can be covered as if $z = 2(\chi + 1)$ for $d = 2$. There are several classes which satisfy this condition, *i.e.*, the renormalization group (RG) solution for momentum-conserving systems,[16]

$$(\chi, z, \beta) = \left(1 - \frac{d}{2}, 1 + \frac{d}{2}, \frac{2-d}{2+d}\right); \tag{17}$$

weak-coupling phase,[41]

$$(\chi, z, \beta) = \left(1 - \frac{d}{2}, 2, \frac{2-d}{4}\right); \tag{18}$$

and irrelevant long-range region[42]

$$(\chi, z, \beta) = \left(\frac{d^2-4d+4}{6-4d}, \frac{8-4d-d^2}{6-4d}, \frac{d^2-4d+4}{8-4d-d^2}\right). \tag{19}$$

Furthermore, the exponents $(\chi, z)$ are generally not independent of each other. The most common relation between them is Galilean invariance: $\chi + z = 2$. Substituting Galilean invariance into Eq. (16) yields $\alpha = \frac{2}{z} - 1$ ($z \neq 2$). Upon combining it with $\gamma = \frac{2}{z}$, we unveil $\alpha = \gamma - 1$, while $z = 2$ gives rise to the logarithmic divergence. It implies that Galilean invariant solutions with $z(d = 2) \neq 2$ will violate $\kappa(L)|_{d=2} \sim log\, L$, *i.e.*, $z = \frac{4d}{d+1}$,[43] $z = \frac{2d+1}{d+1}$,[44] and $z = \frac{2d+4}{d+3}$.[45]

## 2.3. Non-equilibrium scheme

The non-equilibrium scheme can be associated with the following velocity structure functions

$$S_q(L) = \langle|J_S(x+L) - J_S(x)|^q\rangle \propto L^{\varsigma_q}. \tag{20}$$

In the absence of multifractality,[46] the exponent $\varsigma_q$ is linear, $\varsigma_q = q\varsigma_1$. For a steady-state problem, $J_Q(x+L) \equiv J_Q(x)$ and one can obtain $\kappa(L) \propto LS_1(L)$. One typical value[47] is $\varsigma_2 = 2(\chi - 1)$, which means

$$\alpha = \varsigma_1 + 1 = \frac{\varsigma_q}{q} + 1 = \chi. \tag{21}$$

$S_1(L)$ is the total entropy output in $[x, x + L]$, which must equal to the total entropy production rate for steady-state. Therefore, the thermal conductivity in terms of the structure functions can be understood as the ability to ensure the entropy generation. In the sense of entropy ensure, the entropic scaling $\kappa(L) \sim LS_1(L)$ can be extended into problems with internal heat sources, and in this situation, the entropy density obeys the quenched Kardar-Parisi-Zhang (QKPZ) equation.[48] For a

wide class of engineering problems like information processing,[49-52] it is entropy erasure that entails heat transfer in essence. Accordingly, the entropic structure function can be regarded as a more physically meaningful scheme for the anomalous thermal conductivity.

In the presence of Galilean invariance, $\alpha = \chi$ becomes $\alpha = 2 - z$, and upon combining it with $\gamma = \frac{2}{z}$, we arrive at $\alpha = 2 - \frac{2}{\gamma}$. The non-equilibrium scaling $\alpha = \chi$ deviates from result by the equilibrium method, and there exists no logarithmic divergence. Coincidence between the equilibrium non-equilibrium scheme requires $z = 2$ or $\chi = 1$. $z = 2$ exhibits the Brownian MSD ($\gamma = 1$), while $\chi = 1$ corresponds to the ballistic regime. Interestingly, if the "transit time" is replaced by the "correlation time" (or "saturation time"), $t_C \sim L^z$, we get $\alpha = 2\chi + z - 2$. Then, Galilean invariance will give rise to a crossover from the equilibrium form in Eq. (16) to the non-equilibrium result in Eq. (21). It illustrates that the Green-Kubo formula with "transit time" truncation is not identical with the non-equilibrium calculation, more precisely, $\eta \neq \alpha$. Indeed, $t_C = \frac{L}{v}$ is not a rigorous law and will be invalid in the absence of a finite sound velocity. A recent counterexample can be found in Ref. [53], wherein usual fluctuating hydrodynamics no longer holds owing to breakdown of time-reversal symmetry. It is shown that in 1D uniformly charged systems, the truncating time is fitted by $t_C \sim L^{1.5 \pm 0.001}$.

In contrast with Ref. [7], the non-equilibrium exponent is suggested $\alpha = 2 - \frac{2}{\gamma}$ rather than $\alpha = \gamma - 1$. This deviation is caused by the different regimes of the continuous-time random walk (CTRW). Nonlinear fluctuating hydrodynamics possesses a integer-order Fokker-Planck equation (FPE), yet the LW model in Ref. [7] is paired with the fractional Fokker-Planck equation (FFPE). The approach via the entropic structure functions is inapplicable in several situations. The first situation is that $\varsigma_q$ does not satisfy the linear growth. Another one is the divergence exceeding ballistic transport, $\alpha > 1$, which corresponds to $\varsigma_q > 0$, i.e., $\varsigma_q = \frac{q}{3}$ in the 1941 theory (K41) of Kolmogorov.[54] That is because $\alpha > 1$ has not been found in existing investigations, which may exist in hyperdiffusive systems.[55] Furthermore, the upper bound of the conductivity or current has been assessed by the stochastic entropy production rate and Cauchy-Schwarz inequality.[56,57] Consequently, the scaling $sup[\kappa(L)] \sim sup[L\mathcal{S}_1(L)]$ must be constrained by the entropic bound.

## 2.4. Non-Kardar-Parisi-Zhang scaling
Although the KPZ universality class predicts $\alpha = \frac{1}{3}$ for 1D systems, recent experimental and numerical investigations show more possible values. Crnjar *et al.* [15] observed $\alpha \approx \frac{1}{2}$ through a computational study on 1D poly-3,4-ethylenedioxythiophene (PEDOT). Nonuniversal heat conduction is likewise observed in the 1D Fermi-Pasta-Ulam-

β (FPU-β) lattices,[58] wherein the divergent exponent $\alpha$ varies from 0.25 to 0.35. In the measurements by Lee *et al.*,[59] a continuous divergence with $0.1 \leq \alpha \leq 0.5$ was reported in 2-μm-to-1-mm-long single-walled carbon nanotubes (SWCNTs). Such deviations from $\alpha(d = 1) = \frac{1}{3}$ entail non-KPZ scaling in anomalous heat conduction, which should be able to expect continuously distributed $\alpha$. This can be realized by generalized KPZ equations such as the class of correlated noises,[42] namely, $\langle \eta_{SV}(\boldsymbol{x}, t) \eta_{SV}(\boldsymbol{x}', t') \rangle \sim$

$$\begin{cases} |\boldsymbol{x} - \boldsymbol{x}'|^{2\rho - d} \delta(t - t'), \frac{d^2 - 2d}{8d - 12} \leq \rho \leq \frac{d+1}{2} \\ \delta(\boldsymbol{x} - \boldsymbol{x}') |t - t'|^{2\theta - 1}, 0.167 < \theta < 0.5 \end{cases} . \quad (22)$$

For the spatially correlated noise, the scaling exponents are given by

$$(\chi, z) = \left( \frac{2\rho - d + 2}{3}, \frac{d - 2\rho + 4}{3} \right), \quad (23)$$

while the temporally correlated noise corresponds to

$$(\chi, z) = \left( 1.69\theta + 0.22, \frac{2\chi + 1}{1 + 2\theta} \right). \quad (24)$$

The correlated noises expect $\frac{1}{3} < \eta(d = 1) \leq 1$ and $\frac{1}{2} \leq \alpha \leq 1$ in the non-equilibrium scheme. Exponents smaller than $\alpha < \frac{1}{3}$ can arise from other models, *i.e.*, the memory,[60] nonlocal,[61] and fractional-order[62] classes. Specially, the solution of the memory model reads an extension of the momentum-conserving case, namely,

$$(\chi, z) = \left( \frac{2 - d}{2}, \frac{2 + d}{2 + 2\phi} \right), \quad (25)$$

with $0 < \phi < 1$. It can cover arbitrary $-\frac{1}{3} < \eta(d = 1) < \frac{1}{3}$, but only allows one non-equilibrium exponent, $\alpha = \frac{1}{2}$.

These modified models can be applied to describing a set of nanoscale systems with distributed $\alpha$. There are four characteristics needed to be taken account of, including the equilibrium time-dependence by Eq. (16), non-equilibrium length-dependence ($\alpha = \chi$), dimensionality-dependence, and MSD. These characteristics correspond to at least three scaling exponents. Hence, a specific model or solution with one or two tunable parameters is testable. Some modified classes such as the temporally correlated noise break Galilean invariance, which will invalidate the scaling laws $\alpha = \gamma - 1$ and $\alpha = 2 - \frac{2}{\gamma}$. This enables the length-dependent thermal conductivity to coexist with the Brownian MSD ($\gamma = 1$). In analogy with "anomalous yet Brownian" diffusion,[63] we shall term this nontrivial argument as "anomalous yet Brownian" heat conduction.

## 3. Conclusion and discussion
We employ entropy production fluctuations in low-dimensional and mesoscopic heat conduction and establish an entropic Burgers-KPZ class. A generic scaling framework is thereafter presented based on the dynamical scaling theory of nonlinear fluctuating hydrodynamics. We show that the scaling laws in the equilibrium and non-equilibrium schemes, $\alpha = \gamma - 1$ and $\alpha = 2 - \frac{2}{\gamma}$, are analytically traceable to

Galilean invariance. Meanwhile, the dimensionality-dependence, logarithmic divergence, failure of the "transit time" truncation, and non-KPZ scaling are expected likewise. For 1D systems, two universal classes are presented: $\alpha = \frac{1}{3}$ in the equilibrium scheme, and $\alpha = \frac{1}{2}$ in the non-equilibrium scheme. Another well-known argument with $\alpha = \frac{2}{5}$ is absent, which can also be derived from stochastic equations of the MCT.[1] A possible reason is that in the MCT, the Langevin equation is obtained for variables expressed by Fourier-mode amplitudes, whereas the Langevin equation in our work describes the stochastic entropy production rate. Some solutions argue that the logarithmic divergence might not be universal in 2D cases, which is consistent with the simulation by Saito and Dhar.[64] It should be clarified that the dimensionality effects appear not only in $\alpha = \alpha(d)$ but also beyond the length-dependence. In a recent study by Miyazaki *et al.*,[65] Eq. (4) is associated with the "anomalous"[66] or "hidden"[67] entropy production rate. The results reveal that the dimensionality will contribute a prefactor $\mathcal{P}(d)$ to the thermal conductivity, namely, $\kappa = \kappa(d, L) \propto \mathcal{P}(d)L^{\alpha(d)}$. It is interesting that hydrodynamic equations are established for heat conduction in solids. Indeed, the similarity between hydrodynamics and heat conduction is traceable to the conservative laws for physical quantities such as the mass, momentum and energy (see Ref. [4,6,17]). Although the local entropic functionals are not conservative, the derivations of their governing equations rely on the conservative laws of local mass and energy, which hold for both hydrodynamics and heat conduction.

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## Conflict of interest
There are no conflicts to declare.

## Supporting information
Not applicable

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